MATHAUDIT / DETAILED REPORT

Correctness, proof completeness, and novelty audit

Convergence Rate of Birkhoff Average for Toral Quasi-Periodic Rotations and Applications

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Prompt	Standardized prompt
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Paper license	arXiv non-exclusive distribution license
Scope	Statements, proofs, and novelty only

Important limitation. AI-generated analysis. This report is not a correctness certificate or a substitute for expert scrutiny. A Correct status may coexist with yellow typos or minor formal corrections that do not affect substantive soundness. It means that this audit found no unresolved substantive error under the stated criteria; it is not formal verification and should remain open to expert challenge and correction.

SUMMARY

01	Statements	Correct
02	Proofs	Contains incorrect or incomplete proofs
03	Novelty	No non-novelty findings

01 / STATEMENTS

Statements

Correct

The dyadic Besov estimates, their sharpness examples, the discrete analogue, and the statistical-stability theorem are correct. The Hamilton–Jacobi rates are also verified with the regularity actually used by their proof: the two occurrences of $W^{1,1}$ must be $W^{1,\infty}$, and the constant must be allowed to depend on the relevant norm of the potential. This is a local hypothesis correction and does not change the rates or the paper's overall mathematical conclusion.

Theorem 1.1 and Corollary 1.3: Continuous and discrete Birkhoff-average upper bounds

Pages 4–6 and 11–25 · Theorem 1.1, Corollary 1.3, and Sections 3.1 and 3.3 · arXiv:2608.02489v2

Correct

On each dyadic shell, the Diophantine condition separates the values $k \cdot \omega$ by $C_\omega 2^{-j\sigma}$. Lemma 3.1 therefore gives the required ℓ^p small-divisor bound. Hausdorff–Young and the Besov shell norm then yield T^{-1} above the threshold; splitting at $2^N \asymp T^{1/\sigma}$ gives $T^{-s/\sigma}$ below it and the critical factor $(\log T)^{1-1/q}$. The suspension argument for the discrete rotation has the same shell estimates with the additional integer frequency and proves Corollary 1.3.

Theorem 1.2: Sharpness and near-sharpness examples

Pages 4–5 and 15–23 · Theorem 1.2 and Section 3.2 · arXiv:2608.02489v2

Correct

A smooth one-mode coboundary already attains a nonzero multiple of T^{-1} along a sequence. For badly approximable two-dimensional frequencies, the continued-fraction construction contributes a positive amount from each odd convergent and produces the critical $\log T/T$ lower bound while remaining in $B^1_{\infty,\infty}$. A superlacunary subsequence of exact-index resonances yields the $T^{-s/\sigma}$ example. Finally, the large partial quotients supplied almost surely by Borel–Bernstein give the $T^{-\alpha}$ subsequential lower bound, while Theorem 1.1 supplies the printed upper bound. Two proof repairs for parts (i) and (ii) are recorded under Proofs.

Theorem 1.4: Hamilton–Jacobi homogenization rates

Pages 6 and 25–32 · Theorem 1.4 and Section 4 · arXiv:2608.02489v2

Minor formal correction

The proof uses $\mathrm{Lip}(u_0)$ and asserts that $x \mapsto \sqrt{\mu + f(x)}$ is uniformly 1/2-Hölder. Those properties follow from $u_0 \in W^{1,\infty}(\mathbb{R})$ and $f \in W^{1,\infty}(\mathbb{T}^n)$, not from the printed $W^{1,1}$ assumptions when $n \geq 2$. Replacing the two displayed $W^{1,1}$ classes by $W^{1,\infty}$ makes the proof valid and leaves every exponent unchanged. The constant in (1.18)–(1.19) must also depend on $\|f\|_{W^{1,\infty}}$ or the corresponding C^2 data; the proof already has this dependence. Under these locally corrected hypotheses, the Birkhoff bounds for the correctors and the convexity estimate for $\overline{H}$ give exactly the stated lower and upper rates.

Theorem 1.5: Statistical regularity of invariant measures

Pages 7 and 32–35 · Theorem 1.5 and Section 5 · arXiv:2608.02489v2

Correct

Invariance permits time averaging under the perturbed flow. Its lift stays within $C|\delta|t$ of the unperturbed rotation, while Theorem 1.1 bounds the unperturbed Lipschitz average by $T^{-1/\sigma}$, or by $T^{-1}\log T$ at $\sigma = 1$. Optimizing in T gives (1.22). For the lower examples, rational approximants produce periodic-orbit measures whose Wasserstein distance from Lebesgue measure is comparable to Q_j^{-1}; the approximation exponent converts this to (1.23). The sequence is realized by the single Lipschitz family $V(x, \delta) = (1, \vartheta - \delta)$ after choosing approximants from one sign.

02 / PROOFS

Proofs

Contains incorrect or incomplete proofs

The main upper-bound mechanism and both applications are correct after local repairs. The written proof of the smooth T^{-1} sharpness statement does not establish its simultaneous-all-regularities and fixed-start formulation, and Proposition 3.7 uses an invalid common exponential model for continued-fraction denominators. Both defects have short verified repairs. The Hamilton–Jacobi section also contains the displayed Sobolev-index mismatch and two mechanically determined signs.

Lemma 3.1 and Propositions 3.2–3.3: Dyadic small-divisor estimates

Pages 11–15 · Section 3.1 · arXiv:2608.02489v2

Correct and complete

The proof correctly uses separation of distinct shell frequencies, Hausdorff–Young for $1 < p \leq 2$, and the high/low shell split. The critical logarithmic exponent is $1/q' = 1 - 1/q$, and the endpoint $q = 1$ is covered by the same sequence estimates. The low-frequency block contains only finitely many modes and is harmless.

Theorem 1.2(i) and Proposition 3.4: The printed construction does not prove the exact quantified claim

Pages 4 and 15–16 · Theorem 1.2(i), Lemma 3.6, and Proposition 3.4 · arXiv:2608.02489v2

Incorrect as written

The theorem asks for one function lying in $B^s_{\infty,\infty}$ for every $s > 0$ and a lower bound along the fixed orbit in the statement. Proposition 3.4 instead constructs a different lacunary function for each selected $s = k + \alpha$, and Lemma 3.6 proves a supremum over starting points. Repair classification: verified repair. Since nonresonance gives $\omega_1 \neq 0$, take the single smooth mean-zero function $f(x) = \cos(2\pi x_1)$. Then

$$\frac{1}{T}\int_0^T f(\omega t)\,dt = \frac{\sin(2\pi\omega_1 T)}{2\pi\omega_1 T}.$$

Choosing T_j so the sine has absolute value one proves the exact fixed-start C/T_j lower bound, and this f belongs to every positive Besov class.

Proposition 3.7, Equation (3.10): Badly approximable denominators need not follow one common exponential base

Pages 16–18 · proof of Proposition 3.7 · arXiv:2608.02489v2

Incorrect as written

The proof asserts $C_1\rho^k \leq Q_k \leq C_2\rho^k$ with one ρ for the convergent denominators of every badly approximable number. Bounded partial quotients give exponential upper and lower bounds, but not generally with the same base: long blocks of different bounded partial quotients can change the local growth rate. Repair classification: verified repair. The only two uses need weaker standard facts. The recurrence gives $Q_{k+2\ell} \geq 2^\ell Q_k$, which uniformly bounds the number of odd convergents in a dyadic shell; bounded partial quotients give $Q_N \leq (A+1)^N$, which yields $N \geq c\log Q_N$. Substituting these two estimates leaves the $\log T_N/T_N$ conclusion unchanged.

Proof of Theorem 1.4: The regularity used is Lipschitz, not $W^{1,1}$

Pages 26–29 · proof of Theorem 1.4 · arXiv:2608.02489v2

Minor formal correction

The estimates use $\mathrm{Lip}(u_0)$ explicitly and use $|f(x) - f(y)| \leq C|x - y|$ to place $\sqrt{\mu + f}$ uniformly in $C^{0,1/2}$. The replacement $W^{1,1} \to W^{1,\infty}$ described under Statements is therefore required throughout this branch of the proof. With that replacement, the corrector estimates (4.10)–(4.18), the action comparison, and the final optimization are valid.

Equations (4.19)–(4.20) and the negative-energy comparison: The energy relation has two reversed signs

Pages 28–29 · Steps 1–2 of the proof of Theorem 1.4 · arXiv:2608.02489v2

Typo

Equation (4.2) gives $|\dot{\eta}|^2/2 - V(\eta) = r$, so (4.19)–(4.20) must read $\dot{\eta} = \pm\sqrt{2(r + V(\eta))}$, not $\pm\sqrt{2(r - V(\eta))}$. Likewise the displayed action of η_0^+ contains $\sqrt{-2V(x)}$ although $V \geq 0$; it must be $\sqrt{2V(x)}$. The surrounding equations (4.3), (4.5), (4.6), and (4.21) all use the plus sign, so the corrections are mechanical and no estimate changes.

Proof of Theorem 1.5: Time averaging, optimization, and rational-orbit lower example

Pages 32–35 · Section 5 · arXiv:2608.02489v2

Correct and complete

The proof correctly separates orbit perturbation from the unperturbed Birkhoff discrepancy and optimizes the time horizon. The distance-to-orbit test function has Lipschitz norm one and integral $1/(4\sqrt{P_j^2 + Q_j^2})$, giving the claimed Wasserstein lower bound. The printed sequence of constant fields extends directly to a single parameter-Lipschitz family, so no additional assumption is needed.

03 / NOVELTY

Novelty

No non-novelty findings

No non-novelty findings.

METHOD

Scope and decision rules

This audit addressed only the correctness of central statements, the correctness and completeness of proofs, and evidence-backed non-novelty. It did not evaluate importance, exposition, style, suitability for publication, or author reputation.

An adverse finding is included only where the report can give a direct calculation, counterexample, formal contradiction, or a verifiable source establishing the precise claim. An inability to verify is kept distinct from a demonstrated defect. Technical compression is not treated as incompleteness when the omitted reasoning is recoverable and correct.

Before any finding was recorded, the argument was closed under immediate, mechanically checkable consequences of facts already established. An observation was suppressed when the printed conclusion followed by elementary logic, direct substitution, weakening, inclusion, or verified monotonicity and no substantive proof obligation or textual correction remained.

Typographical errors and minor formal corrections remain fully visible in yellow with their locations and repairs. When such a defect is unambiguous, local, and harmless to the paper's substantive conclusions, it does not lower the overall statements or proofs status from Correct.

The novelty panel reports only central results for which a verifiable primary source predating the earliest public version that contains the result proves the same statement up to notation, or from which the statement follows directly and immediately. It contains no positive novelty certifications and no entries based merely on an unsuccessful literature search.

All mathematical expressions in this report are compiled by LaTeX. The web presentation independently validates and renders the corresponding expressions with KaTeX.

Sources checked

1. Source paper: arXiv:2608.02489v2

arXiv acknowledgement

Thank you to arXiv for use of its open access interoperability. This service was not reviewed or approved by, nor does it necessarily express or reflect the policies or opinions of, arXiv.

APPENDIX

Standardized prompt

You are a mathematical auditor. Evaluate only the correctness of the paper's central statements, the correctness and completeness of its proofs, and evidence-backed non-novelty. Do not make editorial comments or evaluate importance, exposition, style, or suitability for publication. Keep the report neutral, precise, and professional.

PART 1 — CORRECTNESS OF THE MAIN STATEMENTS

Identify the paper's central statements and assess each one separately using exactly one of these outcomes:

- Correct: the statement has a correct, verifiable proof; or, despite a defect in the printed proof, you can give a fully verified repair or independent proof.
- Incorrect: you can give strong formal evidence that the statement as written is false, preferably a counterexample or a complete contradiction.
- Not able to verify: the supplied proof does not establish the statement, and you can neither supply a fully verified proof nor disprove it.

For every adverse or unverified outcome, restate the exact claim and hypotheses, identify its location in the reviewed version, and give the decisive reasoning. Do not infer that a theorem is false merely because its printed proof is defective.

Assign the overall statements status by the following priority:

1. Contains wrong statements, if at least one central statement is Incorrect.
2. Contains unsupported statements, if no central statement is Incorrect but at least one is Not able to verify.
3. Correct, if every central statement is Correct, apart from any typographical or minor formal findings allowed by Part 4.

PART 2 — CORRECTNESS AND COMPLETENESS OF THE PROOFS

Audit the proofs of the central statements and every material internal or cited input on which they depend. For each proof or proof segment, use the narrowest supported outcome:

- Correct and complete: every substantive step is verifiable and no case or crucial nontrivial step is missing.
- Incorrect as written: a displayed calculation, inference, lemma, or cited application is formally false or inapplicable.
- Incomplete as written: a necessary substantive step or case is missing, even though no explicit contradiction has been proved.
- Not able to verify: the available paper and sources are insufficient to decide whether the step is valid.

Do not flag omitted routine details, compressed arguments, or "analogously" steps when the omitted reasoning is recoverable and correct. For every defect, identify its smallest exact location, explain its downstream dependency, and classify any proposed repair as one of:

- Verified repair: supply enough reasoning to verify that the repair works throughout every affected use.
- Plausible repair only: describe the possible repair but do not treat it as established.
- No repair supplied.

Assign the overall proofs status by the following priority:

1. Contains incorrect or incomplete proofs, if at least one proof is Incorrect as written or Incomplete as written.
2. Contains unverified proofs, if there is no proved incorrectness or incompleteness but at least one material proof is Not able to verify.
3. Correct, if all audited proofs are Correct and complete, apart from any typographical or minor formal findings allowed by Part 4.

PART 3 — NOVELTY OF THE MAIN STATEMENTS

This is a one-sided, evidence-only check. Evaluate only central results. Report a result only when a verifiable primary source, publicly available before the earliest public version of the paper that contains the result, proves the same statement up to a change of notation, or when the result follows directly and immediately from such an earlier theorem.

For each non-novelty finding, provide the earlier source, its public date, the exact theorem or proposition, and an explicit hypothesis-and-conclusion mapping. A later source cannot establish non-novelty. Do not flag a result merely because an easier proof exists or because it follows through a nontrivial argument. Do not flag a result that the paper itself explicitly presents as known, a survey item, or an alternative proof.

Mention only statements that meet this non-novelty threshold. Say nothing about statements for which no such evidence was found: do not certify novelty, list unsuccessful searches, or add statement-level "no prior source found" entries. If the list is empty, the section may say only "No non-novelty findings."

PART 4 — TECHNICAL AND EVIDENTIARY RULES

1. Criticism must never be overstated. Call a statement incorrect, a proof incorrect or incomplete, or a result non-new only when strong formal evidence supports that exact judgment. Otherwise use Not able to verify where that outcome exists, or remain silent in the novelty section.

2. Review a specified paper version. Give exact page, theorem, equation, or proof locations; restate relevant quantifiers and hypotheses; and check domains, endpoints, parameter ranges, constants, and dependencies explicitly. Apply a defect only to the regimes and downstream results that actually depend on it.

3. Immediate-consequence closure: when deciding whether a conclusion is supported, treat the argument as including every immediate formal consequence of facts established earlier in the paper and available at that point. An immediate formal consequence is a local, mechanically checkable inference using only definitions already in force, elementary logic, direct substitution or renaming, weakening of an inequality or set inclusion, or a monotonicity or symmetry property explicit in a definition or already proved.

4. Such an inference must introduce no new lemma, estimate, construction, case analysis, limiting or compactness argument, external result, or new issue of uniformity or parameter dependence. Check the direction of monotonicity and the dependence of every threshold or constant explicitly. Do not exchange pointwise and uniform conclusions, change exceptional sets, interchange quantifiers or limits, or extend an endpoint under this rule unless the required fact was already established.

5. If the claimed conclusion follows after immediate-consequence closure, it is fully supported. Do not report a gap, unsupported statement, minor formal correction, or recommendation to add an explanatory sentence, and do not mention the suppressed item in the report. This suppression does not apply to a genuine typographical error as defined below, because a typo is a literal notation defect that still requires a textual correction. For example, if $P(delta)$ is proved for every sufficiently small delta > 0 and P is immediately monotone in the required direction, then the claim that $P(delta)$ holds for every delta > 0 requires no correction; the omitted monotonicity step is exposition rather than an audit finding.

6. Immediate-consequence closure permits reconstruction of an omitted inference; it does not permit silent alteration of the printed domain, hypotheses, or quantifiers of an intermediate statement later invoked outside its stated scope. If formal validity requires changing the text of that statement or its stated range, report the appropriate finding even when the repair is easy. Thus a result stated only for $t > 0$ and subsequently applied at $t = 0$ remains a formal scope mismatch unless the endpoint case was separately established before the application.

7. Apply this mandatory gate before recording any adverse, unverified, typographical, or minor formal finding: (a) identify the exact assertion or inference alleged to be defective; (b) reconstruct the strongest conclusion following from the preceding argument after all admissible immediate formal consequences are added; (c) identify the specific defect that remains — a formally demonstrated falsehood, a concrete nontrivial unresolved proof obligation, or a genuine domain, hypothesis, quantifier, or notation mismatch requiring a textual correction; and (d) state the evidence and minimal repair. If no specific defect remains, or if the only possible addition would be explanatory, suppress the finding entirely. Use Not able to verify only when a precise, nontrivial unresolved obligation can be identified.

8. A typographical error is an incorrect symbol, index, exponent, sign, variable name, or notation mismatch whose unique intended correction is mechanically determined from the surrounding text and does not alter the mathematical argument. Report it in yellow as Typo, with the printed text, correction, exact location, and reason it is harmless. It does not lower an overall status.

9. A minor formal correction is a boundary, endpoint, or displayed parameter-range defect that can be repaired consistently in every affected statement by a local change, without a new argument, without invalidating any related result or application, and without creating another substantive defect. Report it in yellow as Minor formal correction, with the same details. It does not lower an overall status.

10. Do not use either minor category if more than one materially different repair is possible; if the repair changes the substance or scope of a central result; if an excluded case is used later; if the correction fails to propagate consistently; or if a new mathematical idea is required. In such cases use the strongest substantive classification actually supported by evidence.

11. Write every mathematical expression as valid, compilable LaTeX. Use ... for inline mathematics and

...

for displayed mathematics, with balanced delimiters and braces. Do not leave raw LaTeX commands outside math delimiters.

12. Separate theorem truth from the correctness of the printed proof. A theorem can be Correct in Part 1 because of a fully verified independent proof while its printed proof is adverse in Part 2. Conversely, a defective proof alone never licenses an Incorrect theorem classification.

Organize the report under Parts 1–3 and apply Part 4 throughout. Do not add editorial commentary, scores, rankings, recommendations, or a list of suppressed observations.

END OF REPORT

Each claim in this prototype report remains open to independent challenge and revision.