MATHAUDIT / DETAILED REPORT

Correctness, proof completeness, and novelty audit

First Families of Regular Polygons and their Mutations

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Paper	arXiv:1612.09295v15
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Audit date	August 18, 2026
Prompt	Standardized prompt
Novelty cutoff	December 29, 2016
Paper license	arXiv non-exclusive distribution license
Scope	Statements, proofs, and novelty only

Important limitation. AI-generated analysis. This report is not a correctness certificate or a substitute for expert scrutiny. A Correct status may coexist with yellow typos or minor formal corrections that do not affect substantive soundness. It means that this audit found no unresolved substantive error under the stated criteria; it is not formal verification and should remain open to expert challenge and correction.

SUMMARY

01	Statements	Contains unsupported statements
02	Proofs	Contains incorrect or incomplete proofs
03	Novelty	No non-novelty findings

01 / STATEMENTS

Statements

Contains unsupported statements

The exact star-point geometry, the First Family height formulas, the cyclotomic scaling-field basis, and the periodic orbit of each $S[k]$ center are correct. The manuscript appropriately labels its mutation, edge, step-sequence, and invariance programs as conjectures, and they are not treated as false claims here. Two theorem-level upgrades are not able to be verified: the text does not prove that the entire proposed $S[k]$ polygons and their generalized families survive as tiles of the limiting web, and the final assertion that every regular N-gon has exactly $C_N = \varphi(N)/2$ realized dual pairs depends on the stated Invariance Conjecture and is followed only by an invalid chirality/Jacobian argument.

Theorem 2.1 and Lemma 3.1: First Family construction and scaling

Pages 13–14 and 19 · First Family Theorem and First Family Scaling Lemma · arXiv:1612.09295v15

Correct

For even N, reflection of the tangent function gives $s_{N/2-k} = 1/s_k$. Placing the center a half-side from star$[k]$ therefore gives

$$\frac{h_{S[k]}}{h_N} = s_1 s_k = \frac{s_1}{s_{N/2-k}},$$

and the two specified star points determine the conforming polygon. For odd N, passing to the conforming $2N$-gon doubles the indices and yields $h_{S[k]}/h_N = s_2 s_{2k}$ in the $2N$ notation. Dividing the $k = 1$ formula by the general formula gives $h_{S[1]}/h_{S[k]} = s_1/s_k = \mathrm{scale}[k]$. Apart from the small-$N$ wording correction reported below, these identities establish the geometric family claimed in the theorem.

Lemma 3.3 and Lemma 3.4: Primitive scales form a unit basis of the maximal real cyclotomic subfield

Pages 20–22 · Scaling Field Lemma and Alternate Generators Lemma · arXiv:1612.09295v15

Correct

Writing $\zeta = e^{2\pi i/N}$ gives

$$\frac{\tan(\pi/N)}{\tan(k\pi/N)} = \frac{\zeta - 1}{\zeta + 1}\frac{\zeta^k + 1}{\zeta^k - 1}.$$

For $\gcd(k, N) = 1$, both this element and its inverse are algebraic integers because the relevant factors are quotients of Galois-conjugate cyclotomic integers. There are $\varphi(N)/2$ such scales with $1 \le k < N/2$. Dividing a rational dependence among them by s_1 would give a dependence among the primitive cotangent values, which are linearly independent; hence the scales are a basis of $\mathbb{Q}(\zeta + \zeta^{-1})$. The Möbius relation between $\tan^2(\pi/N)$ and $2\cos(2\pi/N)$ also proves that the former generates the same real field.

Verifiable source: Girstmair, Some linear relations between values of trigonometric functions at kπ/n

Lemma 4.1: The centers of the First Family polygons have the stated periodic orbits

Pages 25–26 · Canonical orbits of $S[k]$ centers · arXiv:1612.09295v15

Correct

The similarity called Du sends the midpoint polygon of N to N and sends the edge orbit of $\mathrm{star}[k]$ to a vertex-supported outer-billiards orbit. The star-polygon orbit advances by k vertices and has length $N/\gcd(k, N)$. Substitution of the First Family height formula identifies $Du(\mathrm{star}[k])$ with the center $cS[k]$, so that center has exactly the asserted itinerary and period. This statement concerns the center; it does not by itself prove that every point of the proposed polygon shares the itinerary.

Web-preservation claim for the $S[k]$ and generalized families: A center orbit does not establish survival of an entire polygonal tile

Pages 24–32 · opening of Section 4, Lemma 4.1, Evolution of the Web Parts I–II, and use of the GFFT · arXiv:1612.09295v15

Not able to verify

The section announces that every First Family $S[k]$ exists in the complement of the limiting singularity set and that its effective star points support the GFFT family. Lemma 4.1 proves only the itinerary of $cS[k]$. To promote this to a tile, one must show that every point in the proposed polygon remains in the same sequence of continuity domains and that no iterate of its interior meets a singular ray. The subsequent 'shear and rotate' discussion argues from one-sided continuity that a portion of each base segment survives, but gives no domain inequalities over a full cycle and no induction identifying the surviving portion with all edges of the displayed $S[k]$. Part II itself says the step evolution 'should imply' the asserted surviving star points. Thus the full-tile and generalized-family conclusions remain unsupported, although the center-orbit theorem is valid.

Final unnumbered theorem: The count of realized dual pairs is conditional on the Invariance Conjecture

Page 80 · theorem beginning 'Every regular N-gon will have C_N distinct dual pairs' · arXiv:1612.09295v15

Not able to verify

The arithmetic set of primitive indices has cardinality $C_N = \varphi(N)/2$, but the theorem concerns realized adjacent $S[k]$–$S[k^*]$ pairs in the web. The manuscript's abstract explicitly says that the Invariance Conjecture implies this count, and Appendix II introduces the relevant reflection behavior as an assumption before giving extensive examples and tables. No argument proves for arbitrary N that every primitive index produces an actual adjacent pair, or that no further pair occurs. The final paragraph's chirality and Jacobian assertions do not fill that obligation and are formally false as explained in the proofs findings. The statement should remain a conjecture or be made conditional on the Invariance Conjecture unless an independent existence-and-exhaustion proof is supplied.

02 / PROOFS

Proofs

Contains incorrect or incomplete proofs

The trigonometric and cyclotomic proofs in Sections 1–3 and the center-orbit proof in Lemma 4.1 are correct. The arguments that extend center dynamics to full polygonal tiles and the symbolic calculations selected from approximate itineraries are incomplete. The proposed proof of the final dual-pair theorem is incorrect: a regular polygon has reflection symmetry, the dihedral relation is $fr = r^{-1}f$, and every continuity branch of outer billiards has Jacobian determinant 1, not 0. Several local wording errors have uniquely determined repairs and are reported separately without changing the overall statements status.

Sections 1–3: Star-point, First Family, and scaling-field calculations

Pages 7–23 · Lemmas 1.1–3.5 and Theorem 2.1 · arXiv:1612.09295v15

Correct and complete

The scaling identity is the direct tangent quotient

$$\frac{\tan(k\pi/N)}{\tan(jk\pi/N)} = \frac{\text{scale}_N[jk]}{\text{scale}_N[k]},$$

the First Family formulas follow from complementary tangent indices, and the cyclotomic-unit argument plus primitive-cotangent independence supplies the required basis cardinality and independence. The small domain and labeling qualifications reported below are local and do not alter these core derivations.

GFFT and web-survival argument: The proof constructs candidate polygons but not their claimed dynamical survival

Pages 16–18 and 27–32 · Lemma 2.3 and Evolution of the Web Parts I–II · arXiv:1612.09295v15

Incomplete as written

Lemma 2.3 correctly computes the height and location of a candidate conforming polygon once a star point of $S[k]$ is declared effective. It does not prove which star points survive the singularity web. The later argument assumes the local web advances by the proposed step and then infers that the same arithmetic progression of star points survives; it does not prove that the intervening segments avoid all competing outer-billiards domains. A complete repair needs explicit half-plane/domain inequalities for every step of the cycle, followed by an induction showing that the full claimed boundary—and not merely a nonempty subsegment—survives.

Lemma 4.2: Affine equivariance is conjugacy, not commutation

Page 26 · proof of the Twice-Odd Lemma · arXiv:1612.09295v15

Incorrect as written · verified local repair

The proof says that T commutes with the outer-billiards map merely because both are affine. Arbitrary affine maps do not commute. The intended conclusion is nevertheless obtained by affine equivariance: if $P' = T(P)$, support lines and midpoints are preserved, so

$$T \circ \tau_P = \tau_{P'} \circ T.$$

Once the manuscript's displayed T is checked to send the embedded $N/2$ polygon to the corresponding N polygon, this conjugacy maps the local singularity webs. Replacing commutation by this conjugacy argument repairs the lemma locally.

Appendix I exact calculations: Approximate surrogate orbits do not certify the symbolic itinerary

Pages 48–53 · Example A1, parts (i)–(iii) · arXiv:1612.09295v15

Incomplete as written

The exact formulas for the candidate M_x parameters are reconstructed from symbolic sequences first obtained by iterating 8- or 30-decimal approximations. For a discontinuous piecewise isometry, a single sign or atom error changes the exact reconstruction. The manuscript says errors are easy to detect and that survival is clear by inspection, but supplies neither interval enclosures nor exact signed-distance bounds proving that every one of the 150, 140, or 600 iterates stays in the selected atom. The algebra after fixing an itinerary is exact; the itinerary itself is not rigorously certified. A repair is to propagate rational or algebraic interval bounds and verify a nonzero margin from every singular boundary at every listed iterate.

Proof following the final dual-pair theorem: The chirality and zero-Jacobian argument contradicts the map's defining formula

Pages 79–80 · paragraphs preceding and following the final theorem · arXiv:1612.09295v15

Incorrect as written

A regular unmarked N-gon is carried to its reflected copy by an isometry already contained in its dihedral symmetry group, so clockwise and counterclockwise drawings do not provide the asserted geometric chirality. The dihedral relation is $fr = r^{-1}f$, not $f \circ r = -r \circ f$. More decisively, Definition 4.1 gives each continuity branch as $\tau(p) = 2v - p$, whose derivative is $-I_2$ and whose Jacobian determinant is 1, not 0. Thus the claimed failure of smoothness does not imply the existence of any dual pair, much less exactly C_N pairs. Repair classification: no repair of the theorem is supplied; the manuscript would need a separate existence-and-exhaustion argument.

Lemma 1.3 and Theorem 2.1: The star-point labels and the small-N range must be stated

Pages 8 and 13–14 · Two-Star Lemma and First Family Theorem · arXiv:1612.09295v15

Minor formal correction

Two coordinates determine height through $h = (p_2 - p_1)/(s_k \pm s_j)$ only when their indices j, k and their same-side/opposite-side relation are known, as the proof itself states; two unlabeled points admit several interpretations. Also, for $N = 3$ or 4 there is no second distinct star point with which $S[1]$ can be 'strongly conforming' under Definition 2.1. State the Two-Star Lemma for two labeled star points and either restrict the strong-conformity clause of Theorem 2.1 to $N \geq 5$ or explicitly exempt the degenerate small cases. These local changes leave the height formulas unchanged.

Corollary 3.1: The printed index range includes an undefined tangent

Page 22 · Corollary 3.1 · arXiv:1612.09295v15

Minor formal correction

The statement says every s_j with $j < N$ is a rational linear combination of primitive s_k. When N is even, $j = N/2$ is included but $s_{N/2} = \tan(\pi/2)$ is undefined. Replace the range by $1 \leq j < N/2$; values on the other side follow from tangent symmetry where defined. The field-basis proof then applies exactly as written.

Discussion following the Invariance Conjecture: 'Algebraic integer' should read 'algebraic unit'

Page 64 · comparison of s_1^2 with $\lambda_N = 2\cos(2\pi/N)$ · arXiv:1612.09295v15

Typo

The manuscript says the traditional generator λ_N may fail to be an algebraic integer. In fact $\lambda_N = \zeta_N + \zeta_N^{-1}$ is always an algebraic integer. The surrounding comparison is about selecting a unit generator, and λ_N need not be a unit. Replacing 'integer' by 'unit' gives the uniquely consistent statement and does not affect any proof.

03 / NOVELTY

Novelty

No non-novelty findings

No non-novelty findings.

METHOD

Scope and decision rules

This audit addressed only the correctness of central statements, the correctness and completeness of proofs, and evidence-backed non-novelty. It did not evaluate importance, exposition, style, suitability for publication, or author reputation.

An adverse finding is included only where the report can give a direct calculation, counterexample, formal contradiction, or a verifiable source establishing the precise claim. An inability to verify is kept distinct from a demonstrated defect. Technical compression is not treated as incompleteness when the omitted reasoning is recoverable and correct.

Before any finding was recorded, the argument was closed under immediate, mechanically checkable consequences of facts already established. An observation was suppressed when the printed conclusion followed by elementary logic, direct substitution, weakening, inclusion, or verified monotonicity and no substantive proof obligation or textual correction remained.

Typographical errors and minor formal corrections remain fully visible in yellow with their locations and repairs. When such a defect is unambiguous, local, and harmless to the paper's substantive conclusions, it does not lower the overall statements or proofs status from Correct.

The novelty panel reports only central results for which a verifiable primary source predating the earliest public version that contains the result proves the same statement up to notation, or from which the statement follows directly and immediately. It contains no positive novelty certifications and no entries based merely on an unsuccessful literature search.

All mathematical expressions in this report are compiled by LaTeX. The web presentation independently validates and renders the corresponding expressions with KaTeX.

Sources checked

1. Source paper: arXiv:1612.09295v15
2. Girstmair, Some linear relations between values of trigonometric functions at $k\pi/n$

arXiv acknowledgement

Thank you to arXiv for use of its open access interoperability. This service was not reviewed or approved by, nor does it necessarily express or reflect the policies or opinions of, arXiv.

APPENDIX

Standardized prompt

You are a mathematical auditor. Evaluate only the correctness of the paper's central statements, the correctness and completeness of its proofs, and evidence-backed non-novelty. Do not make editorial comments or evaluate importance, exposition, style, or suitability for publication. Keep the report neutral, precise, and professional.

PART 1 — CORRECTNESS OF THE MAIN STATEMENTS

Identify the paper's central statements and assess each one separately using exactly one of these outcomes:

- *Correct: the statement has a correct, verifiable proof; or, despite a defect in the printed proof, you can give a fully verified repair or independent proof.*
- *Incorrect: you can give strong formal evidence that the statement as written is false, preferably a counterexample or a complete contradiction.*
- *Not able to verify: the supplied proof does not establish the statement, and you can neither supply a fully verified proof nor disprove it.*

For every adverse or unverified outcome, restate the exact claim and hypotheses, identify its location in the reviewed version, and give the decisive reasoning. Do not infer that a theorem is false merely because its printed proof is defective.

Assign the overall statements status by the following priority:

1. *Contains wrong statements, if at least one central statement is Incorrect.*
2. *Contains unsupported statements, if no central statement is Incorrect but at least one is Not able to verify.*
3. *Correct, if every central statement is Correct, apart from any typographical or minor formal findings allowed by Part 4.*

PART 2 — CORRECTNESS AND COMPLETENESS OF THE PROOFS

Audit the proofs of the central statements and every material internal or cited input on which they depend. For each proof or proof segment, use the narrowest supported outcome:

- *Correct and complete: every substantive step is verifiable and no case or crucial nontrivial step is missing.*
- *Incorrect as written: a displayed calculation, inference, lemma, or cited application is formally false or inapplicable.*
- *Incomplete as written: a necessary substantive step or case is missing, even though no explicit contradiction has been proved.*
- *Not able to verify: the available paper and sources are insufficient to decide whether the step is valid.*

Do not flag omitted routine details, compressed arguments, or "analogously" steps when the omitted reasoning is recoverable and correct. For every defect, identify its smallest exact location, explain its downstream dependency, and classify any proposed repair as one of:

- *Verified repair: supply enough reasoning to verify that the repair works throughout every affected use.*
- *Plausible repair only: describe the possible repair but do not treat it as established.*
- *No repair supplied.*

Assign the overall proofs status by the following priority:

1. *Contains incorrect or incomplete proofs, if at least one proof is Incorrect as written or Incomplete as written.*
2. *Contains unverified proofs, if there is no proved incorrectness or incompleteness but at least one material proof is Not able to verify.*
3. *Correct, if all audited proofs are Correct and complete, apart from any typographical or minor formal findings allowed by Part 4.*

PART 3 — NOVELTY OF THE MAIN STATEMENTS

This is a one-sided, evidence-only check. Evaluate only central results. Report a result only when a verifiable primary source, publicly available before the earliest public version of the paper that contains the result, proves the same statement up to a change of notation, or when the result follows directly and immediately from such an earlier theorem.

For each non-novelty finding, provide the earlier source, its public date, the exact theorem or proposition, and an explicit hypothesis-and-conclusion mapping. A later source cannot establish non-novelty. Do not flag a result merely because an easier proof exists or because it follows through a nontrivial argument. Do not flag a result that the paper itself explicitly presents as known, a survey item, or an alternative proof.

Mention only statements that meet this non-novelty threshold. Say nothing about statements for which no such evidence was found: do not certify novelty, list unsuccessful searches, or add statement-level "no prior source found" entries. If the list is empty, the section may say only "No non-novelty findings."

PART 4 — TECHNICAL AND EVIDENTIARY RULES

1. Criticism must never be overstated. Call a statement incorrect, a proof incorrect or incomplete, or a result non-new only when strong formal evidence supports that exact judgment. Otherwise use Not able to verify where that outcome exists, or remain silent in the novelty section.

2. Review a specified paper version. Give exact page, theorem, equation, or proof locations; restate relevant quantifiers and hypotheses; and check domains, endpoints, parameter ranges, constants, and dependencies explicitly. Apply a defect only to the regimes and downstream results that actually depend on it.

3. Immediate-consequence closure: when deciding whether a conclusion is supported, treat the argument as including every immediate formal consequence of facts established earlier in the paper and available at that point. An immediate formal consequence is a local, mechanically checkable inference using only definitions already in force, elementary logic, direct substitution or renaming, weakening of an inequality or set inclusion, or a monotonicity or symmetry property explicit in a definition or already proved.

4. Such an inference must introduce no new lemma, estimate, construction, case analysis, limiting or compactness argument, external result, or new issue of uniformity or parameter dependence. Check the direction of monotonicity and the dependence of every threshold or constant explicitly. Do not exchange pointwise and uniform conclusions, change exceptional sets, interchange quantifiers or limits, or extend an endpoint under this rule unless the required fact was already established.

5. If the claimed conclusion follows after immediate-consequence closure, it is fully supported. Do not report a gap, unsupported statement, minor formal correction, or recommendation to add an explanatory sentence, and do not mention the suppressed item in the report. This suppression does not apply to a genuine typographical error as defined below, because a typo is a literal notation defect that still requires a textual correction. For example, if $P(delta)$ is proved for every sufficiently small delta > 0 and P is immediately monotone in the required direction, then the claim that $P(delta)$ holds for every delta > 0 requires no correction; the omitted monotonicity step is exposition rather than an audit finding.

6. Immediate-consequence closure permits reconstruction of an omitted inference; it does not permit silent alteration of the printed domain, hypotheses, or quantifiers of an intermediate statement later invoked outside its stated scope. If formal validity requires changing the text of that statement or its stated range, report the appropriate finding even when the repair is easy. Thus a result stated only for $t > 0$ and subsequently applied at $t = 0$ remains a formal scope mismatch unless the endpoint case was separately established before the application.

7. Apply this mandatory gate before recording any adverse, unverified, typographical, or minor formal finding: (a) identify the exact assertion or inference alleged to be defective; (b) reconstruct the strongest conclusion following from the preceding argument after all admissible immediate formal consequences are added; (c) identify the specific defect that remains — a formally demonstrated falsehood, a concrete nontrivial unresolved proof obligation, or a genuine domain, hypothesis, quantifier, or notation mismatch requiring a textual correction; and (d) state the evidence and minimal repair. If no specific defect remains, or if the only possible addition would be explanatory, suppress the finding entirely. Use Not able to verify only when a precise, nontrivial unresolved obligation can be identified.

8. A typographical error is an incorrect symbol, index, exponent, sign, variable name, or notation mismatch whose unique intended correction is mechanically determined from the surrounding text and does not alter the mathematical argument. Report it in yellow as Typo, with the printed text, correction, exact location, and reason it is harmless. It does not lower an overall status.

9. A minor formal correction is a boundary, endpoint, or displayed parameter-range defect that can be repaired consistently in every affected statement by a local change, without a new argument, without invalidating any related result or application, and without creating another substantive defect. Report it in yellow as Minor formal correction, with the same details. It does not lower an overall status.

10. Do not use either minor category if more than one materially different repair is possible; if the repair changes the substance or scope of a central result; if an excluded case is used later; if the correction fails to propagate consistently; or if a new mathematical idea is required. In such cases use the strongest substantive classification actually supported by evidence.

11. Write every mathematical expression as valid, compilable LaTeX. Use ... for inline mathematics and

...

for displayed mathematics, with balanced delimiters and braces. Do not leave raw LaTeX commands outside math delimiters.

12. Separate theorem truth from the correctness of the printed proof. A theorem can be Correct in Part 1 because of a fully verified independent proof while its printed proof is adverse in Part 2. Conversely, a defective proof alone never licenses an Incorrect theorem classification.

Organize the report under Parts 1–3 and apply Part 4 throughout. Do not add editorial commentary, scores, rankings, recommendations, or a list of suppressed observations.

END OF REPORT

Each claim in this prototype report remains open to independent challenge and revision.